The hypoelliptic Laplacian

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For Professor А.Н. Паршин

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- Ellipticity stable property by small deformation.

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- The operator of Kolmogorov model of hypoelliptic operators studied by Hörmander.

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3. Deformation connects objects of analysis to geometric objects.
4. It connects spectral invariants to closed geodesics, like in Selberg's trace formula.

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- $\mathcal{L}_b$ Fokker-Planck operator.

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- $\int_{\mathbf{R}} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}} = 1.$
- $\int_{\mathbf{R}} e^{iy\xi - y^2/2} \frac{dy}{\sqrt{2\pi}} = e^{-\xi^2/2}.$

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- Fourier + analyticity.

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- Translation $\simeq$ conjugation.

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- Origin of rigidity is cohomological.
- When $b \to 0$, only $\mathrm{Sp}\left(-\Delta^{S^1}/2\right)$ survives.
- When $b \to +\infty$, $L_b \simeq \frac{1}{2}y^2 - y\frac{\partial}{\partial x}$.

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- By making $b \to +\infty$, we get Poisson's formula par interpolation.

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- $\exp\left(t\Delta^X/2\right)$ heat operator on $C^\infty(X, \mathbf{R})$.
- $\exp\left(t\Delta^X/2\right)$ can be considered as an element g of a semigroup acting on $C^\infty(X, \mathbf{R})$.

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- Beware: This cohomology is now infinite dimensional.

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- Analogue of formula of McKean-Singer $\chi(g) = \mathrm{Tr_s}[g \exp(-D_R^2/2b^2)] \ldots$

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$$\mathrm{Tr}^{C^\infty(X,\mathbf{R})}[g] = \mathrm{Tr_s}^R\left[g\exp\left(-D_R^2/2b^2\right)\right].$$

- Analogue of formula of McKean-Singer $\chi(g) = \mathrm{Tr_s}\left[g\exp\left(-D_R^2/2b^2\right)\right]\ldots$
- ...used in proof of Lefschetz fixed point formulas of Atiyah-Bott.

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- $\left(S^\cdot\left(T^*X \oplus N^*\right) \otimes \Lambda^\cdot\left(T^*X \oplus N^*\right), d^{TX\oplus N}\right)$= fibrewise algebraic de Rham complex.

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Making $b \to 0$, $\mathcal{L}_b^X$ deforms $\frac{1}{2}\left(-\Delta^X + c\right)$.

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- This identity splits as an identity of semisimple orbital integrals.

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- For $t > 0$, $\mathrm{Tr}^{[\gamma]}\left[\exp\left(-t\left(C^{\mathfrak{g},X} + c\right)/2\right)\right]$ orbital integral on adjoint orbit of γ:

$$\int_{Z(\gamma)\backslash G} \mathrm{Tr}^E \left[p_t\left(g^{-1}\gamma g\right)\right] dg.$$

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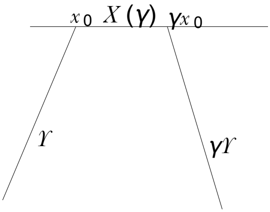

$d(Y, \gamma Y) \geq C(1+ \quad |Y|)$

A fundamental identity

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- $A''_b = \overline{\partial}^{\mathcal{X}} + i_y/b^2$ acts on $\Omega^{(0,\cdot)}\left(\mathcal{X}, \pi^*\left(\Lambda^{\cdot}\left(T^*X\right) \otimes F\right)\right)$.

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- If pure Serre duality was used, the 'Laplacian' would be 0!

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- This still fails for the above hypoelliptic Hodge theory.
- This works for a Hodge theory, in which the Kähler form ω^X is replaced by $|Y|^2_{g^{\widehat{TX}}} \omega^X$.

RRG and Fourier transform

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- The theory of the hypoelliptic Laplacian is an attempt to invert the Fourier transform.

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- Introducing the geodesic flow is a way of forcing Fourier transform in the analysis.

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- The hypoelliptic Laplacian introduces extra degrees of freedom in fibre direction…
- …which exist even in very rigid situations.

J.-M. Bismut, *Hypoelliptic Laplacian and Bott-Chern cohomology*, Preprint (Orsay) (2011).

______, *Hypoelliptic Laplacian and orbital integrals*, Annals of Mathematics Studies, vol. 177, Princeton University Press, Princeton, NJ, 2011. MR 2828080

______, *Index theory and the hypoelliptic Laplacian*, Proceedings of the Conference in honour of Jeff Cheeger (Xianzhe Dai and Xiaochun Rong, eds.), Birkhäuser Boston Inc., 2012, pp. 181–232.